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η–ξ spacetime and the tilde freedom in thermo field dynamics*

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Abstract. The CM operations are introduced in the theory of η–ξ spacetime. They have the same properties as the tilde rules in thermo field dynamics (TFD); we can, therefore, identify them with the tilde operation. This leads to the conclusion that the tilde system in TFD can be regarded as the system in region II (mirror universe) of η–ξ spacetime.

1. Introduction

In recent years thermo field dynamics (TFD) [1, 2], as a real-time formulation of finite-temperature field theory, has attracted much interest. Evolving from condensed matter physics, it has been applied to high-energy physics, cosmology, nuclear physics, etc. In the framework of TFD, the tilde fields and the tilde conjugation rules play important roles. The tilde fields denote the degrees of freedom which are introduced in TFD to double the system. This doubling of the degrees of freedom characterizes the thermal properties of the system. The tilde rules show the connection between the tilde fields and non-tilde fields, so it can be regarded as definition of the tilde operation in TFD. The tilde rules have the following forms [2, 3]:

$$
\begin{aligned}
&[AB]^{\tilde{}} = \tilde{A}\tilde{B} \qquad && [C_1 A + C_2 B]^{\tilde{}} = C_1^* \tilde{A} + C_2^* \tilde{B} \\
&(\tilde{A})^{\tilde{}} = \sigma A \qquad && (\tilde{A})^{+} = (A^{+})^{\tilde{}}
\end{aligned}
\tag{1.1}
$$

where $\sigma = 1$ for bosons and -1 for fermions. In this paper we will discuss the physical meaning of the tilde fields and derive the tilde rules in the framework of the theory of η–ξ spacetime [4–8]. We find that the tilde fields are fields in region II (mirror universe) in η–ξ spacetime. The tilde operation corresponds to the following operations: first the fields in region I (our universe) are mapped onto the fields in region II, then a charge conjugation is operated on the obtained fields to get the tilde fields. Starting from these operations, we get the tilde rules.

The field theory in η–ξ spacetime is a geometrical formalism for finite temperature field theory. η–ξ spacetime is a maximal analytical complex extension of the manifold with topology $S^1 \times R^3$. Both the imaginary- and the real-time formalisms of finite-temperature

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field theory can be connected in η–ξ spacetime, which can be regarded as the background spacetime for finite-temperature field theory. In previous papers [4–9], we have proven the following results. The field theory on the Euclidean section in η–ξ spacetime corresponds to the imaginary-time finite-temperature field theory, the field theory on the Lorentzian section in η–ξ spacetime corresponds to TFD. On the Euclidean section, quantum fields automatically satisfy the periodicity for imaginary-time τ, so the Euclidean Green functions in η–ξ spacetime correspond to the imaginary-time thermal Green functions. On the Lorentzian section, 'horizons' lead to the doubling of degrees of freedom of fields, and the vacuum Green functions in η–ξ spacetime are equal to the real-time thermal Green functions with a 2×2 matrix in Minkowskian spacetime; and TFD corresponds to the Hamiltonian formalism of η–ξ field theory. Similar to the Hawking radiation theory for black holes, the vacuum for fields in η–ξ spacetime is found to correspond to the thermal state for a static observer in Minkowskian spacetime, and the η–ξ vacuum has the same form as the thermal vacuum in TFD. All these factors lead to an interesting connection between η–ξ spacetime and finite-temperature field theory.

This paper is organized as follows. In section 2, the geometrical and topological structure of η–ξ spacetime is briefly reviewed. We put emphasis on the coordinate relations between regions I and II on the Lorentzian section. In section 3, starting from the coordinate relations, we construct the CM operations in η–ξ spacetime. Here the C operation stands for the charge conjugation. The M operation stands for the mapping of fields caused by the inversion of coordinates η and ξ. In the Minkowskian description, the inversions of η and ξ correspond to a finite translation of time in the imaginary-time direction. Some properties of the CM operation are presented and found to be the same as the tilde rules. In section 4, the CM operation is applied to the discussion of some important relations in TFD. It is suggested that the tilde freedoms can be regarded as the freedoms in region II on the Lorentzian section in η–ξ spacetime. Section 5 is devoted to the conclusion.

2. η–ξ spacetime

η–ξ spacetime is a four complex dimensional spacetime with complex metric:

$$ds^2 = \alpha^{-2}(\xi^2 - \eta^2)^{-1}(-d\eta^2 + d\xi^2) + dy^2 + dz^2 \tag{2.1}$$

where α is a real constant, and η, ξ, y, z are complex variables. While ξ, y, z are limited to real variables, and η is a pure imaginary variable $i\sigma$, the Euclidean section of η–ξ spacetime is obtained:

$$ds^2 = \alpha^{-2}(\xi^2 + \sigma^2)^{-1}(d\sigma^2 + d\xi^2) + dy^2 + dz^2. \tag{2.2}$$

The metric (2.2) is singular at $\sigma = \xi = 0$, so it describes a Euclidean spacetime with topology $S^1 \times R^3$. It can be regarded as background spacetime for imaginary-time finite-temperature field theory. The periodicity of imaginary-time now becomes the periodicity of polar angle $\theta = \alpha\tau$, where $\alpha = 2\pi/\beta$.

The Lorentzian section of η–ξ spacetime is

$$ds^2 = \alpha^{-2}(\xi^2 - \eta^2)^{-1}(-d\eta^2 + d\xi^2) + dy^2 + dz^2 \tag{2.3}$$

but it should be noted that (2.3) is gained from (2.1) by limiting η, ξ, y, z to real variables. According to (2.3), the singularities of the Lorentzian section are described by

$$\eta \pm \xi = 0 \tag{2.4}$$

which divide the Lorentzian section into four disjointed parts I, II, III, IV (figure 1). Each of them is identified with a Minkowskian spacetime which can be related to other regions only through complex paths. This can be seen from the following transformations. If a Minkowskian spacetime

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{2.5}$$

is transformed by

$$\eta = \alpha^{-1} e^{\alpha x} \sinh \alpha t \qquad \xi = \alpha^{-1} e^{\alpha x} \cosh \alpha t \tag{2.6a}$$

with y, z unchanged, the result is metric (2.3). But equations (2.6a) can illustrate only a quadrant of the Lorentzian section, i.e. region I. To get other regions, the transformation of Minkowskian spacetime are:

$$\eta = -\alpha^{-1} e^{\alpha x} \sinh \alpha t \qquad \xi = -\alpha^{-1} e^{\alpha x} \cosh \alpha t \tag{2.6b}$$

$$\eta = \alpha^{-1} e^{\alpha x} \cosh \alpha t \qquad \xi = \alpha^{-1} e^{\alpha x} \sinh \alpha t \tag{2.6c}$$

$$\eta = -\alpha^{-1} e^{\alpha x} \cosh \alpha t \qquad \xi = -\alpha^{-1} e^{\alpha x} \sinh \alpha t \tag{2.6d}$$

which lead to regions II, III, IV, respectively. The appearance of singularities (2.4) and the existence of several regions make it possible for the Lorentzian section to provide the doubling of degrees of freedom.

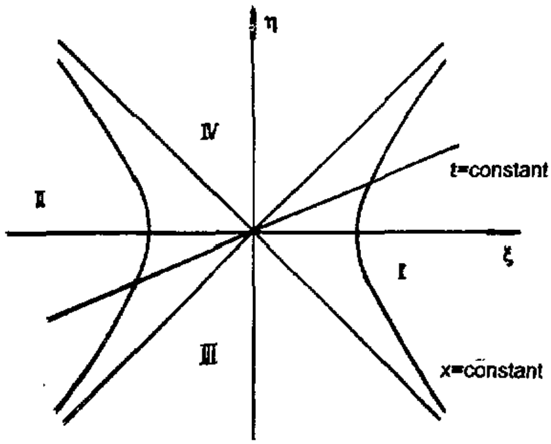

Figure 1.

There are close relations between transformations (2.6a) and (2.6b):

$$\begin{aligned} \eta &= -\alpha^{-1} e^{\alpha x} \sinh \alpha t = \alpha^{-1} e^{\alpha x} \sinh \alpha (t - i\beta/2) \\ \xi &= -\alpha^{-1} e^{\alpha x} \cosh \alpha t = \alpha^{-1} e^{\alpha x} \cosh \alpha (t - i\beta/2). \end{aligned} \tag{2.7}$$

Using Minkowskian coordinates (t, x, y, z), the relations between a point $\alpha_1(t_1, x_1, y_1, z_1)$ in region I and its reflected point $\alpha_2(t_2, x_2, y_2, z_2)$ in region II are:

$$t_2 = t_1 - i\beta/2 \qquad x_2 = x_1 \qquad y_2 = y_1 \qquad z_2 = z_1 \tag{2.8}$$

i.e. their Minkowskian coordinates differ only in an imaginary-time interval $i\beta/2$. While expressed by coordinates (η, ξ, y, z), the relations between $\alpha_1(\eta_1, \xi_1, y_1, z_1)$ and $\alpha_2(\eta_2, \xi_2, y_2, z_2)$ are:

$$\eta_2 = -\eta_1 \qquad \xi_2 = -\xi_1 \qquad y_2 = y_1 \qquad z_2 = z_1. \tag{2.9}$$

Another important relation is that the direction of time t in region II is against the time direction in region I. One can see this by the use of the Killing vectors in η–ξ spacetime. The time-like Killing vectors on the Lorentzian section are defined as [5]

$$\left(\frac{\partial}{\partial\lambda}\right)^{\alpha} = \varepsilon\alpha\left(\xi\frac{\partial}{\partial\eta} + \eta\frac{\partial}{\partial\xi}\right) \tag{2.10}$$

in which ε is 1 and -1 for regions I and II respectively. In region I, the out-normal direction of hyperplane $\lambda =$ constant coincides with that of the equal-time surface $t =$ constant. But in region II, it is anti-parallel to that of surface $t =$ constant.

3. The CM operations in η–ξ spacetime

The relations (2.9) imply that the inversions of coordinates η and ξ will result in the transformation between the physical systems in regions I and II. This situation is a little similar to the CPT operation in Minkowskian spacetime. However, the CM operations, which will be discussed later, result in a new transformation. This can be seen from (2.8) and (2.9). If the coordinates η and ξ are inversed, the Minkowskian space coordinates are kept intact, while the Minkowskian time is translated $\mathrm{i}\beta/2$ in the imaginary-time direction.

First let us consider the M operation. M is the operator which maps a field onto the field with inversed η and ξ coordinates. Similar to the time inversion in Minkowskian spacetime, the M operation is anti-linear. The reason is that the direction of time-like Killing vector is parallel to the direction of Minkowskian time t in region I, but anti-parallel to the Minkowskian time direction in region II. The M operations on the bosonic field $\varphi(\eta, \xi, y, z)$ and fermionic field $\psi(\eta, \xi, y, z)$ are

$$\begin{aligned} M\varphi(\eta, \xi, y, z)M^{-1} &= \varphi(-\eta, -\xi, y, z) \\ M\psi(\eta, \xi, y, z)M^{-1} &= T\psi(-\eta, -\xi, y, z) \end{aligned} \tag{3.1}$$

the matrix T can be determined by the invariance of the Lagrangian of fermionic field in η–ξ spacetime [7]:

$$\mathcal{L} = \bar{\psi}\left[\mathrm{i}\varepsilon\alpha(\xi^2 - \eta^2)^{1/2}\left(\gamma^0\frac{\partial}{\partial\eta} + \gamma^1\frac{\partial}{\partial\xi}\right) + \mathrm{i}\gamma^2\frac{\partial}{\partial y} + \mathrm{i}\gamma^3\frac{\partial}{\partial z} - m + \frac{\mathrm{i}}{2}\varepsilon\alpha\frac{\gamma^0\eta - \gamma^1\xi}{(\xi^2 - \eta^2)^{1/2}}\right]\psi \tag{3.2}$$

under the M operation. Here ε is given in (2.10). Since only operation between regions I and II is considered, the Lagrangian of fields in regions III and IV are omitted. From the above restriction, T is adopted as $\gamma^2\gamma^0\gamma^1$.

Since the Minkowskian coordinates are adopted in the tilde rules, we shall change the M operation into a form with Minkowskian coordinates. For bosonic fields, we can directly change the η–ξ coordinates into the Minkowskian coordinates. For fermionic fields, this can be done by the use of the relations between fermionic fields in η–ξ vierbeins and the same fields in Lorentz vierbeins [7]:

$$\begin{aligned} \psi &\to (\cos\tfrac{1}{2}\alpha\tau - \mathrm{i}\gamma^0\gamma^1\sin\tfrac{1}{2}\alpha\tau)\psi \\ \bar{\psi} &\to \bar{\psi}(\cos\tfrac{1}{2}\alpha\tau + \mathrm{i}\gamma^0\gamma^1\sin\tfrac{1}{2}\alpha\tau). \end{aligned} \tag{3.3}$$

As for region II

$$\tau = \beta/2 = \pi/\alpha \tag{3.4}$$

(3.3) becomes

$$\begin{aligned}&\psi(-\eta,-\xi,y,z)\to -\mathrm{i}\gamma^0\gamma^2\psi(t-\mathrm{i}\beta/2,\chi)\\&\bar{\psi}(-\eta,-\xi,y,z)\to\bar{\psi}(t-\mathrm{i}\beta/2,\chi)\mathrm{i}\gamma^0\gamma^2.\end{aligned}\tag{3.5}$$

To get the tilde rules, the charge conjugation operation is also needed:

$$\begin{aligned}&C\varphi C^{-1}=\varphi^{+}\\&C\psi C^{-1}=\mathrm{i}\gamma^0\gamma^2\bar{\psi}^T.\end{aligned}\tag{3.6}$$

From (3.1), (3.5) and (3.6), it is easy to get the CM operations on bosonic field φ and fermionic field ψ:

$$\begin{aligned}&CM\varphi(t,\chi)M^{-1}C^{-1}=\varphi^{+}(t-\mathrm{i}\beta/2,\chi)\\&CM\psi(t,\chi)M^{-1}C^{-1}=\gamma^0\bar{\psi}^T(t-\mathrm{i}\beta/2,\chi)=\psi^*(t-\mathrm{i}\beta/2,\chi).\end{aligned}\tag{3.7}$$

According to the above discussions, a set of rules identified with (1.1) can be obtained. First we consider bosonic fields φ_1 and φ_2

$$\begin{aligned}&CM[\varphi_1(t_1,\chi_1)\varphi_2(t_2,\chi_2)]M^{-1}C^{-1}=\varphi_1^{+}(t_1-\mathrm{i}\beta/2,\chi_1)\varphi_2^{+}(t_2-\mathrm{i}\beta/2,\chi_2)\\&CM[CM\varphi(t,\chi)M^{-1}C^{-1}]M^{-1}C^{-1}=\varphi(t,\chi)\\&CM[C_1\varphi_1(t_1,\chi_1)+C_2\varphi_2(t_2,\chi)]M^{-1}C^{-1}=C_1^*\varphi_1^{+}(t_1,\chi)+C_2^*\varphi_2^{+}(t_2,\chi_2)\\&CM[\varphi^{+}(t,\chi)]M^{-1}C^{-1}=[CM\varphi(t,\chi)M^{-1}C^{-1}]^{+}.\end{aligned}\tag{3.8}$$

Here the c-numbers are conjugated under the CM operation for the reason that the M operation is anti-linear. If $\varphi^{+}(t-\mathrm{i}\beta/2,\chi)$ and the CM operations are taken as the tilde field $\tilde{\varphi}(t,\chi)$ and tilde operation respectively, one notes immediately that (3.8) are just the tilde conjugation rules for bosonic field.

As in the bosonic field case, we can obtain the following results for fermionic fields ψ_1 and ψ_2:

$$\begin{aligned}&CM[\psi_1(t_1,\chi_1)\psi_2(t_2,\chi_2)]M^{-1}C^{-1}=\psi_1^*(t_1-\mathrm{i}\beta/2,\chi_1)\psi_2^*(t_2-\mathrm{i}\beta/2,\chi_2)\\&CM[\psi^{+}(t,\chi)]M^{-1}C^{-1}=[CM\psi(t,\chi)M^{-1}C^{-1}]^{+}\\&CM[C_1\psi_1(t_1,\chi_1)+C_2\psi_2(t_2,\chi_2)]M^{-1}C^{-1}=C_1^*\psi_1^*(t_1-\mathrm{i}\beta/2,\chi_1)+C_2^*\psi_2^*(t_2-\mathrm{i}\beta/2,\chi_2)\\&CM[CM\psi(t,x)M^{-1}C^{-1}]M^{-1}C^{-1}=\psi(t-\mathrm{i}\beta,\chi)=-\psi(t,\chi).\end{aligned}\tag{3.9}$$

To get the last equation in (3.9), the anti-periodical boundary condition for fermionic field is used.

The properties of the CM operations in η–ξ spacetime are given by (3.8) and (3.9), which coincide with the tilde rules in TFD. Since the tilde operation is defined by (1.1), it can be said that the CM operations in η–ξ spacetime identify with the tilde operation in TFD.

4. Discussion

It is suggested that [10], after the tilde system and thermal vacuum are introduced, the TFD can be constructed by the use of two sets of operators $\{A\}$ and $\{\tilde{A}\}$ with several important relations. Some of these relations are:

(i) At equal time, tilde operators and non-tilde operators commute (for bosonic operators) or anti-commute (for fermionic operators):

$$[A,\tilde{B}]_{\pm}=0\tag{4.1}$$

(ii) There is a set of tilde rules (1.1).

(iii) The thermal vacuum is invariant under the tilde conjugation:

$$[|0(\beta)\rangle]^{\sim} = |0(\beta)\rangle. \tag{4.2}$$

(iv) A spacetime translation of non-tilde operator A is induced by the non-tilde energy–momentum operator P_μ:

$$A(x) = \exp(\mathrm{i}P_\mu \chi^\mu) A \exp(-\mathrm{i}P_\mu \chi^\mu). \tag{4.3}$$

In the last section, we have given the tilde rules through introducing the CM operations in η–ξ spacetime, thus have explained the tilde rules. In this section, we will see whether the other relations can be explained in the theory of η–ξ spacetime.

First let us consider the relation about commutative properties. One remembers that the CM operations map field operators in region I onto corresponding conjugated field operators in region II. According to the conclusion of the last section, the CM operations identify with the tilde operation. Considering these facts, one may regard A and $\tilde{B}$ as operators in regions I and II respectively. As each of the regions is a whole Minkowskian spacetime, no real propagator can relate one region to another. Consequently, no direct causal relation exists between these regions. So the commutation or anti-commutation relations of tilde and non-tilde operators can be explained by the principle of microcausality, which indicates that operators with no causal relations existing between them shall commute or anti-commute. From the point of view of η–ξ spacetime, (4.1) can be generalized to non-equal-time situations.

Now we will show the invariance of thermal vacuum under the CM operations. To do this, one shall know the results of the CM operations on state vectors. Since physical systems in region I are transformed by the CM operations into corresponding systems in region II, the Minkowskian vacuum in region I changes into the Minkowskian vacuum in region II:

$$CM|0\rangle_I = |0\rangle_{II}. \tag{4.4}$$

It is natural to assume that the latter is just the tilde vacuum $|\tilde{0}\rangle$ in TFD. Similarly we can get states $|n\rangle_I(|n\rangle)$ and $|n\rangle_{II}(|\tilde{n}\rangle)$, from which the thermal vacuum is constructed [1]:

$$|0(\beta)\rangle = Z^{-1/2}\sum_n \mathrm{e}^{-\beta En/2}|n\rangle \otimes |\tilde{n}\rangle = Z^{1/2}\sum_n \mathrm{e}^{-\beta En/2}|n\rangle_I \otimes |n\rangle_{II}. \tag{4.5}$$

The second expression is just the vacuum defined on the whole Lorentzian section [5]. The CM operations on (4.5) changes $|n\rangle_I \otimes |n\rangle_{II}$ into $|n\rangle_{II} \otimes |n\rangle_I$. Usually, such an exchange in direct products leads to a different result. But here the two vectors have the same particle number n and the values of n are summed, thus the thermal vacuum does not vary. Hence, if the tilde operation is regarded as the CM operation, we have proved the invariance of thermal vacuum under the tilde operation.

As for the fourth relation, one can find at once that it is the same as the usual translation in Minkowskian spacetime. So it is certainly true in region I which is a Minkowskian spacetime.

The above discussion leads directly to the idea that the non-tilde systems and tilde systems can be regarded as the systems in regions I and II, respectively.

5. Conclusion

In this paper, starting from the coordinate relations in η–ξ spacetime, we introduced the CM operations on the Lorentzian section. Since the CM operations have the same properties as the tilde rules, we identify them with the tilde operation. Furthermore, since the CM operations transform the physical system in region I into the system in region II, it is natural to draw the following conclusion: if the non-tilde system is regarded as the system in region I, the tilde system can be regarded as the system in region II. This conclusion provides a possibility of avoiding the arbitrariness existing in TFD. It may be helpful in understanding the characteristic of doubling the degrees of freedom in TFD.

Finally we shall emphasize again that, although the CM operations are also transformations caused by the inversion of the spacetime coordinates, they are quite different from the CPT operation in Minkowskian spacetime. If Minkowskian coordinates are adopted and imaginary-time is introduced, the CM operations translate the physical systems in the direction of imaginary time, then inverse the gained complex time. Symmetry under such operations is unfamiliar. While we take advantages of η–ξ coordinates, the CM operations have the form of inversions of 'space' coordinate ξ and 'time' coordinate η. Besides, both of the 'space' and 'time' have real coordinates. Thus a new discrete symmetry desired by TFD, i.e. symmetry between tilde and non-tilde systems, is exposed by η–ξ spacetime through symmetry of 'space' and 'time'.

References

[1] Takahashi Y and Umezawa H 1975 *Collect. Phenom.* **2** 55

[2] Umezawa H, Matsumoto H and Tachiki M 1981 *Thermo Field Dynamics and Condensed States* (Amsterdam: North-Holland)

[3] Matsumoto H, Nakano Y and Umezawa H 1983 *Phys. Rev.* D **28** 1931

[4] Gui Y X 1988 *Sci. Sin.* A **31** 1104

[5] Gui Y X 1990 *Phys. Rev.* D **42** 1988

[6] Gui Y X 1992 *Phys. Rev.* D **45** 697

[7] Gui Y X 1992 *Phys. Rev.* D **46** 1869

[8] Gui Y X 1993 *Sci. Sin.* A **36** 561

[9] Xu F and Gui Y X 1988 *Chin. Phys. Lett.* **5** 381

[10] Matsumoto H 1986 *Progress in Quantum Field Theory* ed H Ezawa and S Kamefuchi (Amsterdam: North-Holland)